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Nova methodus quantitates integrales determinandi

Leonhard Euler

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NOVA METHODVS

QVANTITATES INTEGRALES DETERMINANDI.

Auctore

L. E V L E R O.

§. I.

Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant diuisae, veluti $\frac{p dz}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integram simplicissimam huius generis $\int \frac{dz}{lz}$ attinet, facile patet, si eam ita integrari concipiam, vt evanescat posito $z = 0$, tum vero statuatur $z = 1$, quantitatem infinite magnam esse prodituram; quod si enim variabilis z iam proxime ad unitatem accesserit, vt sit $z = 1 - u$, existente u quantitate infinite parua, tum ob $dz = -du$ et $lz = l(1 - u) = -u$ haec formula erit $\int \frac{du}{u}$, cuius valor vtique fit infinitus. At vero dantur omnino huiusmodi formulae integrales $\int \frac{p dz}{lz}$, quae, etiamsi ponatur $z = 1$, tamen valores finitae magnitudinis fortiuntur: quod determinasse

nasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores inuestigandi.

§. 2. Consideremus exempli gratia hanc formulam satis simplicem $\int \frac{(z-1) dz}{1z}$, quae memorata lege integrata valorem finitum habere facile ostendi potest: Posito enim $\frac{z-1}{1z} = y$, vt formula nostra fiat $\int y dz$, ideoque exprimat aream curuae, pro abscissa z applicatam habentis $= y$, ista area a termino $z = 0$ vsque ad terminum $z = 1$ extensa vti-que valorem finitum non multo maiorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z = 0$, fiet etiam applicata $y = 0$, at sumta $z = 1$ pro applicata $y = \frac{z-1}{1z}$ tam numerator quam denominator euanes-
cit, ergo eorum loco substitutis suis differentialibus fiet $y = z = 1$. Pro abscissis autem mediis ponamus $z = e^{-n}$, existente e numero, cuius logarith-
mus hyperbolicus est vnitas, erit

$$y = \frac{e^{-n} - 1}{-n} = \frac{e^n - 1}{n e^n},$$

quae, si n fuerit numerus valde magnus, vt abscissa z fiat minima, applicata erit proxime $y = \frac{1}{n}$; qui ergo valor multo maior erit quam abscissa z ; forma scilicet huius curuae similis erit figurae adiectae, vbi AP denotat abscissam z et PM applicatam y , abscissae vero $AB = 1$ respondet applicata $BC = 1$ qua curua descripta eius area $AMCB$ non multum superabit aream trianguli ABC quae est $= \frac{1}{2}$. Tab. I. Fig. 1.

§. 3. Nuper autem, in aliis inuestigationibus occupatus, praeter expectationem inueni, hanc aream aequalem esse logarithmo hyperbolico binarii, ita vt ea per fractiones decimales fit $l2 = 0,6931471805$; sequenti autem ratiocinio huc sum perductus: Cum reuera fit $lz = \frac{z^0 - 1}{0}$, quia differentiando vtrunque prodit $\frac{dz}{z} = \frac{dz}{z}$, et sumto $z = 1$ vtraque expressio evanescit, loco 0 scribo $\frac{1}{i}$, denotante i numerum infinitum, eritque $lz = i(z^{\frac{1}{i}} - 1)$, hincque applicata

$$y = \frac{z - 1}{i(z^{\frac{1}{i}} - 1)} = \frac{1 - z}{i(1 - z^{\frac{1}{i}})},$$

et formula integralis $\int \frac{(1 - z) dz}{i(1 - z^{\frac{1}{i}})}$. Nunc igitur statu

tuo $z^{\frac{1}{i}} = x$, vt fiat $z = x^i$, vbi notetur pro vtroque integratione termino $z = 0$ et $z = 1$ etiam fore $x = 0$ et $x = 1$; quia igitur hinc fit $dz = ix^{i-1} dx$, formula integralis euadit

$$\int \frac{x^{i-1} dx (1 - x^i)}{(1 - x)},$$

quam ergo integrari oportet a termino $x = 0$ vsque ad terminum $x = 1$.

§. 4. Spectemus nunc i vt numerum valde magnum, et fractio $\frac{1 - x^i}{1 - x}$ resoluitur in hanc progressionem geometricam

$$1 + x$$

$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \dots x^{i-1}$
 cuius singuli termini in $x^{i-1} dx$ ducti et integrati
 praebent hanc seriem

$$\frac{x^i}{i} + \frac{x^{i+1}}{i+1} + \frac{x^{i+2}}{i+2} + \frac{x^{i+3}}{i+3} + \dots + \frac{x^{2i-1}}{2i-1}$$

quae utique euanesceat facto $x = 0$. Nunc igitur su-
 matur $x = 1$ et valor quaesitus nostrae formulae in-
 tegralis erit

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \dots + \frac{1}{2i-1}$$

ubi quidem littera i denotat numerum infinite ma-
 gnum, ita ut numerus horum terminorum sit re-
 vera infinitus. Nihilominus, quia singuli
 termini sunt infinite parui, haec series summam
 habebit finitam, quam sequenti modo ad seriem or-
 dinariam reducere licet.

§. 5. Series inuenta spectari potest tanquam
 differentia inter binas sequentes progressionem har-
 monicas

$$A = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{2i-1}$$

$$B = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots + \frac{1}{i-1}$$

quandoquidem differentia $A - B$ ipsam seriem in-
 ventam exhibet; quia autem numerus terminorum
 seriei A est $2i - 1$, seriei vero $B = i - 1$, ille du-
 plo maior est quam hic, quocirca, ut seriem re-
 gularem obtineamus, singulos terminos seriei B per
 saltum a seriei A termino secundo, quarto, sexto,

1 3

octavo

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octauo etc. auferamus, quo pacto simul ad finem vtriusque peruenietur, eritque

$$A - B = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{8} + \text{etc.}$$

in infinitum, cuius ergo valor est $1/2$, ita vt nunc quidem solide sit demonstratum, formulae integralis propositae $\int \frac{(z-1) dz}{lz}$ casu $z = 1$ valorem reuera esse $= 1/2$.

§. 6. Simile ratiocinium etiam ad formulam integralem generaliore $\int \frac{(z^m - 1) dz}{lz}$ accommodari potest, ac tandem reperietur casu $z = 1$ eius valorem fore $l(m+1)$; quia igitur pari modo erit

$$\int \frac{(z^n - 1) dz}{lz} = l(n+1),$$

si hanc ab illa subtrahamus, prodit sequens integratio

$$\int \frac{(z^m - z^n) dz}{lz} = l \frac{m+1}{n+1}$$

si scilicet integratio a termino $z = 0$ vsque ad terminum $z = 1$ extendatur.

§. 7. Quia autem haec demonstratio per quantitates infinitas et infinite paruas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem inuestigatio maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variables inuoluentium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eodem prin-

principio quoque integrationes hic exhibitae derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes, aliis methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

Lemma I.

§. 8. Si P fuerit functio quaecunque duarum variabilium z et u , ac ponatur $\int P dz = S$, vt etiam S sit functio binarum variabilium z et u , tum erit

$$\int dz \left(\frac{dP}{du} \right) = \left(\frac{dS}{du} \right).$$

Demonstratio.

Cum in integratione formulae $\int P dz$ sola z vt variabilis spectetur, erit $\left(\frac{dS}{dz} \right) = P$, quae formula denuo differentiata sola u pro variabili habita praebet $\left(\frac{d}{du} \frac{dS}{dz} \right) = \left(\frac{dP}{du} \right)$, quae in dz ducta et integrata producit $\left(\frac{dS}{du} \right) = \int dz \left(\frac{dP}{du} \right)$ quandoquidem ex principiis calculi integralis est

$$\int dz \left(\frac{d}{du} \frac{dS}{dz} \right) = \left(\frac{dS}{du} \right) \text{ q. e. d.}$$

Corollarium I.

§. 9. Eodem modo per huiusmodi differentia- lia vbi tantum u pro variabili spectatur ulterius pro- gredi licet, vnde sequentes oriuntur integrationes

$$\left(\frac{d}{du^2} \frac{dS}{dz} \right) = \int dz \left(\frac{d}{du^2} \frac{dP}{dz} \right) \text{ et}$$

$$\left(\frac{d^2 S}{du^2} \right) = \int dz \left(\frac{d^2 P}{du^2} \right)$$

etc.

etc.

Corol-

Corollarium 2.

§. 10. Quod si ergo formula $\int P dz$ fuerit integrabilis, ita ut eius integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int dz \left(\frac{dP}{du} \right), \int dz \left(\frac{d}{du} \frac{dP}{du} \right), \int dz \left(\frac{d^2 P}{du^2} \right) \text{ etc.}$$

integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

Scholion.

§. 11. Ex his quidem formulis si in genere tractentur, parum utilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, ut integrale $\int P dz$, casu saltem particulari, quo post integrationem variabili z certus quidam valor puta $z = a$ tribuitur, commode exhiberi potest, ut hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integrationes ponatur $z = a$, atque hinc ad eiusmodi integrationes plerumque peruenitur, quas aliis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur

Primum principium integrationum.

§. 12. Si P eiusmodi fuerit functio binarum variabilium z et u , ut valor integralis $\int P dz$ saltem casu certo $z = a$ commode exprimi queat, qui valor sit $= S$, functio scilicet ipsius u tantum; tum etiam sequentia integralia si quidem post integrationem

tionem pariter statuatur $z = a$ commode exhiberi poterunt scilicet

$$\int P dz = S$$

$$\int dz \left(\frac{dP}{du} \right) = \left(\frac{dS}{du} \right)$$

$$\int dz \left(\frac{d^2 P}{du^2} \right) = \left(\frac{d^2 S}{du^2} \right)$$

$$\int dz \left(\frac{d^3 P}{du^3} \right) = \left(\frac{d^3 S}{du^3} \right)$$

$$\int dz \left(\frac{d^4 P}{du^4} \right) = \left(\frac{d^4 S}{du^4} \right)$$

etc. etc.

Exemplum I.

§. 13. Si fuerit $P = z^u$, erit quidem in genere

$$\int P dz = \frac{z^{u+1}}{u+1};$$

vnde casu $z = 1$ hic valor satis simplex nascitur $\frac{1}{u+1}$, ita vt fit $S = \frac{1}{u+1}$, cum deinde per differentiationes continuas, dum sola u pro variabili habetur, prodeat

$$\left(\frac{dP}{du} \right) = z^u l z, \text{ tum vero } \left(\frac{d^2 P}{du^2} \right) = z^u (l z)^2, \text{ porro}$$

$$\left(\frac{d^3 P}{du^3} \right) = z^u (l z)^3, \left(\frac{d^4 P}{du^4} \right) = z^u (l z)^4, \text{ etc.}$$

hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z = 1$

$$\int z^u dz = \frac{1}{u+1}$$

$$\int z^u dz l z = - \frac{1}{(u+1)^2}$$

$$\int z^u dz (l z)^2 = + \frac{1 \cdot 2}{(u+1)^3}$$

$$\int z^u dz (l z)^3 = - \frac{1 \cdot 2 \cdot 3}{(u+1)^4}$$

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$$\int z^u dz (l z)^4 = + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(u+1)^5}$$

$$\int z^u dz (l z)^5 = - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(u+1)^6}$$

$$\int z^u dz (l z)^6 = + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{(u+1)^7}$$

$$\int z^u dz (l z)^7 = - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{(u+1)^8}$$

K

vnde

vnde concludimus generaliter fore

$$\int z^u dz (lz)^n = \pm \frac{1.2.3.4 \dots n}{(u+1)^{n+1}}$$

vbi signum + valet si n fit numerus par, alterum vero - si n fit numerus impar. Hae quidem integrationes iam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumimus: breuiter igitur repetamus eos casus, quos iam nuper expediui.

Exemplum 2.

$$\S. 14. \text{ Si fuerit } P = \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}$$

iam dudum demonstraui, formulae $\int P dz$ valorem integralem casu quo post integrationem ponitur $z=1$,

$$\text{effe } S = \frac{\pi}{2n \cos \frac{\pi u}{2n}}$$

$$\text{hinc ergo cum fit } \left(\frac{dP}{du} \right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \cdot l z$$

$$\text{tum vero } \left(\frac{d^2 P}{du^2} \right) = + \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} (l z)^2$$

$$\text{et } \left(\frac{d^3 P}{du^3} \right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} (l z)^3$$

etc. etc.

ex

ex cognito valore S sequentes nacti sumus integrationes

$$\text{I. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} dz = S = \frac{\pi}{2n \cos \frac{\pi u}{2n}}$$

$$\text{II. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} dz \log z = \left(\frac{dS}{du} \right)$$

$$\text{III. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} dz (\log z)^2 = \left(\frac{d^2 S}{du^2} \right)$$

$$\text{IV. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} dz (\log z)^3 = \left(\frac{d^3 S}{du^3} \right)$$

$$\text{V. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} dz (\log z)^4 = \left(\frac{d^4 S}{du^4} \right)$$

etc.

etc.

Exemplum 3.

$$\S. 15. \text{ Si fuerit } P = \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}}$$

simili modo demonstraui valorem formulae integralis $\int P dz$, casu quo post integrationem ponitur $z = 1$, fore

$$S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z = 1$ fuerunt deductae

$$\text{I. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz = S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n}$$

$$\text{II. } \int \frac{-z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz \log z = \left(\frac{dS}{du} \right)$$

$$\text{III. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} dz (\log z)^2 = \left(\frac{d^2 S}{du^2} \right)$$

$$\text{IV. } \int \frac{-z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz (\log z)^3 = \left(\frac{d^3 S}{du^3} \right)$$

$$\text{V. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} dz (\log z)^4 = \left(\frac{d^4 S}{du^4} \right)$$

etc.

etc.

Scholion.

§. 16. Quo igitur vberiores fructus ex hoc principio expectare queamus, præcipuum negotium huc redit; vt eiusmodi functiones binarum variabilium z et u pro P inuestigemus, ita vt valor formulæ integralis saltem certo quodam casu puta $z=1$ succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione est deductum, ita eodem modo continua integratio ad vsum nostrum accommodari poterit.

Lemma II.

§. 17. Si P fuerit functio duarum variabilium z et u , ac ponatur $\int P dz = S$ vt etiam S sit functio duarum variabilium z et u , tum erit $\int S du = \int d$

$= \int dz \int P du$, vbi in integralibus formulis $\int P du$ et $\int S du$ sola u pro variabili habetur, in formula autem $\int dz \int P du$ sola z .

Demonstratio.

Ponatur $\int S du = V$, vt fit $S = \left(\frac{dV}{du}\right)$, ideoque $\left(\frac{dV}{du}\right) = \int P dz$ eritque $\left(\frac{d dV}{dz du}\right) = P$; vnde, per du multiplicando et integrando, erit $\left(\frac{dV}{dz}\right) = \int P du$, ex quo sequitur $V = \int dz \int P du = \int S du$. q. e. d.

Corollarium 1.

§. 18. Hoc modo etiam integratio repeti potest; vnde orietur talis aequatio $\int du \int S du = \int dz \int du \int P du$; hinc autem plerumque parum utilitatis expectari potest, nisi forte istae integrationes commode succedant.

Corollarium 2.

§. 19. Quod si ergo formula $\int P dz$ fuerit integrabilis, scilicet $= S$ altera hinc deducta $\int dz \int P du$ eatenus tantum integrari poterit, quatenus integrale $\int S du$ integrare licet.

Secundum principium integrationum.

§. 20. Si P eiusmodi fuerit functio duarum variabilium z et u , vt formulae integralis $\int P dz$ valor certo saltem casu puta $z = a$ commode exhiberi queat, ita, vt hoc casu quantitas S fiat functio solius variabilis u ; tum etiam pro eodem casu $z = a$

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huius formulae integralis $\int dz \int P du$ valor assignari poterit, si modo formulam $\int S du$ integrare licuerit.

Exemplum I.

§. 21. Sumamus $P = z^u$, eritque $\int P dz = \frac{z^{u+1}}{u+1}$; quae formula casu $z = 1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int P du = \int z^u du = \frac{z^u}{l z}, \text{ et quia}$$

$$\int S du = l(u+1), \text{ erit } \int \frac{z^u dz}{l z} = l(u+1);$$

si quidem post illam integrationem ponatur $z = 1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^u dz}{l z} = l(u+1) + C;$$

atque hic quidem facile intelligitur, hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^u}{l z}$ posito $z = 1$ fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

Corollarium I.

§. 22. Quoniam autem haec constans C non a variabili u pendet, ea retinebit eundem valorem, quicumque numeri determinati pro u accipiantur. Sumamus

namus igitur primo $u=m$, tum vero etiam $u=n$,
 vt habeamus istos valores

$$\text{I. } \int \frac{z^m dz}{lz} = l(m+1) + C \text{ et}$$

$$\text{II. } \int \frac{z^n dz}{lz} = l(n+1) + C$$

quarum altera ab altera subtracta relinquet istam in-
 tegrationem notatu dignissimam

$$\int \frac{(z^m - z^n) dz}{lz} = l \frac{m+1}{n+1}$$

quemadmodum iam supra ex longe aliis principiis
 demonstraui.

Corollarium. 2.

§. 23. Si ad alteram integrationem ascenda-
 mus, quia est $\int P du = \frac{z^u}{lz}$, erit $\int du \int P du = \frac{z^u}{(lz)^2}$;
 tum vero ob

$\int S du = l(u+1) + C$, erit $\int du \int S du = (u+1)(l(u+1)-1) + Cu + D$
 ficque habebimus

$$\int \frac{z^u dz}{(lz)^2} = (u+1)(l(u+1)-1) + Cu + D$$

vbi constantes C et D non ab u pendent; quare vt
 eas eliminemus tres casus determinatos euoluamus

$$\text{I. } \int \frac{z^m dz}{(lz)^2} = (m+1)l(m+1) - m - 1 + Cm + D$$

II.

$$\text{II. } \int \frac{z^n dz}{(lz)^2} = (n+1)l(n+1) - n - 1 + Cn + D$$

$$\text{III. } \int \frac{z^k dz}{(lz)^2} = (k+1)l(k+1) - k - 1 + Ck + D$$

eritque $\text{I} - \text{III} = (m+1)l(m+1) - (k+1)l(k+1) + k - m + C(m-k)$
 et $\text{II} - \text{III} = (n+1)l(n+1) - (k+1)l(k+1) + k - n + C(n-k)$

hincque deducimus

$$\begin{aligned} (\text{I} - \text{III})(n-k) - (\text{II} - \text{III})(m-k) &= \frac{(m+1)(n-k)l(m+1)}{-(k+1)(n-k)l(k+1) + (k-m)(n-k)} \\ &= \frac{-(n+1)(m-k)l(n+1) - (k-n)(m-k)}{+(k+1)(m-k)lk+1} \end{aligned}$$

atque hinc deducimus sequentem integrationem

$$\begin{aligned} \int \frac{dz ((n-k)z^m - (m-k)z^n + (m-n)z^k)}{(lz)^2} &= \\ &+ (m+1)(n-k)l(m+1) \\ &- (n+1)(m-k)l(n+1) \\ &+ (k+1)(m-n)l(k+1). \end{aligned}$$

Corollarium 3.

§. 14. Operae pretium erit aliquot casus euolvere, vbi quidem numeros m , n et k inter se inaequales accipi conuenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m = 2$, $n = 1$ et $k = 0$ erit

$$\int \frac{(z-1)^2 dz}{(lz)^2} = 3l3 - 4l2 = l\frac{2}{3}$$

II. Sit $m = 3$, $n = 1$ et $k = 0$ eritque

$$\int \frac{(z^3 - 3z + 2) dz}{(lz)^2} = \int dz (z-1)^2 (z+2) = 4l4 - 6l2 = 2l2 = l4$$

III.

III. Sit $m = 3$, $n = 2$ et $k = 0$ et erit

$$\int \frac{(2z^3 - 3zz + 1)dz}{(lz)^2} = \int \frac{dz(z-1)^2(2z+1)}{(lz)^2} = 8/4 - 9/3 = 7/2.$$

IV. Sit $m = 3$, $n = 2$ et $k = 1$ et prodit

$$\int \frac{(z^3 - 2zz + z)dz}{(lz)^2} = \int \frac{zdz(z-1)^2}{(lz)^2} = 4/4 - 6/3 + 2/2 = 1/6.$$

Corollarium 4.

§. 25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod ideo necessario vsu venit, ne valores integralium euadant infiniti. Quia enim denominator $(lz)^2$ euanescit casu $z = 1$, si ponamus $z = 1 - \omega$ existente ω infinite paruo, erit

$$lz = -\omega \text{ et } (lz)^2 = +\omega\omega.$$

Neceffe ergo est vt in numeratore adsit factor, qui casu $z = 1 - \omega$ itidem praebeat $\omega\omega$, quod euenit si ibi factor fuerit $(z-1)^2$.

Scholion.

§. 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z = 1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes, in Corollario secundo inuentae, etiamsi multo magis arduae videantur, tamen ex prioribus ope reductionum cognitarum non difficulter deriuari possunt; id quod pro vnico casu ostendisse sufficiet. Ponamus

$$\int \frac{dz(z-1)^2}{(lz)^2} = \frac{p}{lz} + \int \frac{q dz}{lz} \text{ eritque differentiando}$$

$$\frac{dz(z-1)^2}{(lz)^2} = \frac{dp}{lz} - \frac{p dz}{z(lz)^2} + \frac{q dz}{lz}$$

unde aequatis terminis seorsim vel per $(lz)^2$ vel per lz diuisae habebimusque has duas aequalitates.

$$(z-1)^2 = -\frac{p}{z} \text{ et } dp = -q dz$$

ex quarum priore oritur $p = -z(z-1)^2$, hincque

$$\frac{dp}{dz} = -3z^2 + 4z - 1 \text{ ideoque } q = 3z^2 - 4z + 1$$

ita ut sit

$$\int \frac{dz(z-1)^2}{(lz)^2} = -\frac{z(z-1)^2}{lz} + \int \frac{(3z^2 - 4z + 1) dz}{lz}$$

hic autem prius membrum posito $z=1$ sponte euanescit; posito enim $z=1-\omega$ ut sit $lz=-\omega$, erit $p=-\omega\omega(1-\omega)$ ideoque $\frac{p}{lz}=\omega(1-\omega)=0$ ob $\omega=0$, posterius vero membrum in has partes discerpi potest

$$3 \int \frac{(z^2-z) dz}{lz} - \int \frac{(z-1) dz}{lz}$$

prioris autem partis integrale est $3l\frac{z^2}{2}$, posterioris vero $-1l\frac{z}{2}$; ficque totum hoc integrale erit

$$3l\frac{z^2}{2} - l\frac{z}{2} = 3l\frac{z^2}{2} - 4l\frac{z}{2} = l\frac{3z^2-4z}{2}$$

prorsus uti inuenimus. Hoc igitur modo si in genere statuamus

$$\int \frac{v dz}{(lz)^2} = \frac{p}{lz} + \int \frac{q dz}{lz} \text{ erit differentiando}$$

$$\frac{v dz}{(lz)^2} = \frac{dp}{lz} - \frac{p dz}{z(lz)^2} + \frac{q dz}{lz}$$

unde istae duae fluunt aequalitates

$$p = -vz \text{ et } q = -\frac{dp}{dz}$$

Iam

Iam vt terminus $\frac{p}{z}$ euanescat posito $z = 1$, numerator p factorem habere debet $(z - 1)^2$; qui ergo etiam factor esse debet quantitatis V . Sit igitur

$$V = \frac{U(z-1)^2}{z} \text{ eritque } p = -U(z-1)^2, \text{ vnde fit}$$

$$dp = -dU(z-1)^2 - 2Udz(z-1) = (z-1)(dU(z-1) - 2Udz),$$

hincque fit

$$q dz = (z-1)(2Udz - dU(z-1));$$

quia ergo q factorem habet $z-1$ formula $\int \frac{q dz}{z}$ semper in partes resolui potest, quarum integralia per corollariam primum assignare licet, si modo U fuerit aggregatum ex quocunque potestatibus ipsius z ; vnde sequens deducitur theorema.

Theorema.

§. 27. Si fuerit

$$P = A z^\alpha + B z^\beta + C z^\gamma + D z^\delta + \text{etc.}$$

ita, vt summa coefficientium

$$A + B + C + D \text{ etc.} = 0 \text{ tum erit}$$

$$\int \frac{P dz}{z} = A l(\alpha + 1) + B l(\beta + 1) + C l(\gamma + 1) + D l(\delta + 1)$$

si quidem post integrationem statuatur $z = 1$.

Demonstratio.

Cum hoc ipso casu, quo post integrationem ponitur $z = 1$ fit

$$\int \frac{z^n dz}{z} = l(n + 1) + \Delta$$

denotante Δ illam constantem infinitam integratione ingressam erit.

L 2

Af

Iam

$$A \int \frac{z^\alpha dz}{lz} = A l(\alpha + 1) + A \Delta$$

eodemque modo

$$B \int \frac{z^\xi dz}{lz} = B l(\xi + 1) + B \Delta$$

etc. etc.

si haec integralia omnia in vnam summam colligan-
tur erit

$$(A + B + C + D + \text{etc.}) \Delta = 0,$$

hincque erit integrale quaesitum

$$\int \frac{P dz}{lz} = A l(\alpha + 1) + B l(\xi + 1) + C l(\gamma + 1) + D l(\delta + 1) \text{ etc.}$$

q. e. d.

Corollarium 1.

§. 28. Quia supponimus

$$A + B + C + D + \text{etc.} = 0$$

euidens est formulam

$$P = A z^\alpha + B z^\xi + C z^\gamma + D z^\delta + \text{etc.}$$

factorem habere $z - 1$ quemadmodum iam ante no-
tauimus.

Corollarium 2.

§. 29. Quia est

$$(z - 1)^n = z^n - \frac{n}{1} z^{n-1} + \frac{n(n-1)}{1 \cdot 2} z^{n-2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} z^{n-3} + \text{etc.}$$

hoc valore loco P posito, erit $A = 1$ et $\alpha = n$; deinde

$$B = -\frac{n}{1} \text{ et } \xi = n - 1; \text{ porro } C = \frac{n(n-1)}{1 \cdot 2} \text{ et } \gamma = n - 2; \text{ etc.}$$

hinc

hinc igitur erit

$$\int \frac{(z-1)^n dz}{l z} = l(n+1) - \frac{n}{1} l n + \frac{n \cdot n-1}{1 \cdot 2} l(n-1) - \frac{n \cdot (n-1) \cdot (n-2)}{1 \cdot 2 \cdot 3} l(n-2) \\ + \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3)}{1 \cdot 2 \cdot 3 \cdot 4} l(n-3) + \text{etc.}$$

si modo exponens n fuerit nihilo maior, vel saltem vnitatem non minor, quia alioquin casu $z=1$ fractio $\frac{(z-1)^n}{l z}$ fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita vt sufficiat dummodo sit $n > 0$.

Exemplum 2.

§. 30. Sit $P = \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}}$ erit $\int P dz = \frac{\pi}{2n \cos. \frac{\pi u}{2n}}$

si quidem post integrationem ponatur $z=1$ quem ergo valorem literae S tribuimus. Nunc spectata z vt constante erit

$$\int P du = \frac{1}{1 + z^{2n}} (\int z^{n-u-1} du + \int z^{n+u-1} du)$$

ideoque

$$\int P du = - \frac{z^{n-u-1} + z^{n+u-1}}{(1 + z^{2n}) l z} \text{ vnde fiet}$$

$$\int S du = \int \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}} \frac{dz}{l z};$$

cum igitur sit $\cos. \frac{\pi u}{2n} = \sin. \frac{\pi(n-u)}{2n}$ erit

$$\int S du = \int \frac{\pi du}{2n \sin. \frac{\pi(n-u)}{2n}};$$

L 3

hinc

hinc si ponamus

$$\frac{\pi(n-u)}{2n} = \Phi, \text{ erit } d\Phi = -\frac{\pi du}{2n}, \text{ ideoque}$$

$$\int S du = -\int \frac{d\Phi}{\sin \Phi} = -l \operatorname{tang.} \frac{1}{2} \Phi,$$

quocirca habebimus

$$\int S du = -l \operatorname{tang.} \frac{\pi(n-u)}{4n},$$

ita ut posito post integrationem $z=1$ affecuti simus hanc integrationem

$$\int \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \frac{dz}{z} = -l \operatorname{tang.} \frac{\pi(n-u)}{4n} = +l \operatorname{tang.} \frac{\pi(n+u)}{4n}.$$

Exemplum 3.

$$\S. 31. \text{ Sit } P = \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}} \text{ erit}$$

$$\int P dz = \frac{\pi}{2n} \operatorname{tang.} \frac{\pi u}{2n} = S$$

unde fit

$$\int S du = -l \operatorname{cof.} \frac{\pi u}{2n},$$

hinc cum fit

$$\int P du = -\frac{z^{n-u-1} - z^{n+u-1}}{(1-z^{2n})lz}$$

nanciscimur sequentem integrationem, si quidem integrale a termino $z=0$ vsque ad terminum $z=1$ extendatur

$$\int \frac{z^{n-u-1} + z^{n+u-1}}{1-z^{2n}} \frac{dz}{z} = +l \operatorname{cof.} \frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla iam ante fusius expediui; unde iis magis euoluendis non immoror: sed ad sequens problema progredior.

Pro-

Problema.

Si proponantur hae duae series infinitae

$$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + z^5 \cos. 5u + \text{etc. et}$$

$$Q = z \sin. u + z^2 \sin. 2u + z^3 \sin. 3u + z^4 \sin. 4u + z^5 \sin. 5u + \text{etc.}$$

quae binas variables z et u inuoluunt, inuenire relationes inter formulas integrales $\int \frac{P dz}{z}$, $\int P du$ et $\int \frac{Q dz}{z}$, $\int Q du$ aliasque formulas integrales per continuam integrationem inde natas.

Solutio.

§. 32. Cum utraque series sit recurrens, reperitur per formulas finitas

$$P = \frac{z \cos. u - z z}{1 - 2z \cos. u + z z} \quad \text{et} \quad Q = \frac{z \sin. u}{1 - 2z \cos. u + z z}$$

unde fit

$$\int \frac{P dz}{z} = \int \frac{d z \cos. u - z dz}{1 - 2z \cos. u + z z} = -l \sqrt{1 - 2z \cos. u + z z}$$

$$\text{et} \quad \int Q du = \int \frac{z du \sin. u}{1 - 2z \cos. u + z z} = +l \sqrt{1 - 2z \cos. u + z z}$$

ita ut fit

$$\int \frac{P dz}{z} = -\int Q du; \quad \text{tum vero etiam erit}$$

$$\int \frac{Q dz}{z} = \int \frac{d z \sin. u}{1 - 2z \cos. u + z z} = A \tan g. \frac{z \sin. u}{1 - z \cos. u};$$

at si iste arcus differentietur sumto solo angulo u variabili, erit

$$\frac{1}{du} A \tan g. \frac{z \sin. u}{1 - z \cos. u} = \frac{z \cos. u - z z}{1 - 2z \cos. u + z z}$$

ita ut fit $\int \frac{Q dz}{z} = \int P du$.

§. 33. Verum eadem relationes facilius ex ipsis seriebus deriuantur: Cum enim fit

$$P = z$$

$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u \text{ etc.}$ erit

$$\int \frac{P dz}{z} = \frac{z \cos. u}{1} + \frac{z z \cos. 2u}{2} + \frac{z^3 \cos. 3u}{3} \text{ etc.}$$

$$\text{et } \int P du = \frac{z \sin. u}{1} + \frac{z z \sin. 2u}{2} + \frac{z^3 \sin. 3u}{3} \text{ etc.}$$

et quia est

$Q = z \sin. u + z^2 \sin. 2u + z^3 \sin. 3u + \text{etc.}$ erit

$$\int \frac{Q dz}{z} = \frac{z \sin. u}{1} + \frac{z z \sin. 2u}{2} + \frac{z^3 \sin. 3u}{3} \text{ etc.}$$

$$\text{et } \int Q du = -\frac{z \cos. u}{1} - \frac{z^2 \cos. 2u}{2} - \frac{z^3 \cos. 3u}{3} \text{ etc.}$$

vnde manifestum est fore

$$\int \frac{P dz}{z} = -\int Q du \text{ et } \int \frac{Q dz}{z} = \int P du.$$

§. 34. Quo hoc modo vltcrius progredi liceat
statuamus breuitatis gratia

$$P^I = \frac{z \cos. u}{1} + \frac{z^2 \cos. 2u}{2} + \frac{z^3 \cos. 3u}{3} \text{ etc. et } Q^I = \frac{z \sin. u}{1} + \frac{z^2 \sin. 2u}{2} + \frac{z^3 \sin. 3u}{3} \text{ etc.}$$

$$P^{II} = \frac{z \cos. u}{1^2} + \frac{z^2 \cos. 2u}{2^2} + \frac{z^3 \cos. 3u}{3^2} \text{ etc. et } Q^{II} = \frac{z \sin. u}{1^2} + \frac{z^2 \sin. 2u}{2^2} + \frac{z^3 \sin. 3u}{3^2} \text{ etc.}$$

$$P^{III} = \frac{z \cos. u}{1^3} + \frac{z^2 \cos. 2u}{2^3} + \frac{z^3 \cos. 3u}{3^3} \text{ etc. et } Q^{III} = \frac{z \sin. u}{1^3} + \frac{z^2 \sin. 2u}{2^3} + \frac{z^3 \sin. 3u}{3^3} \text{ etc.}$$

$$P^{IIII} = \frac{z \cos. u}{1^4} + \frac{z^2 \cos. 2u}{2^4} + \frac{z^3 \cos. 3u}{3^4} \text{ etc. et } Q^{IIII} = \frac{z \sin. u}{1^4} + \frac{z^2 \sin. 2u}{2^4} + \frac{z^3 \sin. 3u}{3^4} \text{ etc.}$$

etc.

etc.

etc.

etc.

et hinc comparationes ante inuentae continuabuntur

$$P^I = \int \frac{P dz}{z} = -\int Q du, \quad Q^I = \int \frac{Q dz}{z} = \int P du.$$

$$P^{II} = \int \frac{P^I dz}{z} = -\int Q^I du, \quad Q^{II} = \int \frac{Q^I dz}{z} = \int P^I du.$$

$$P^{III} = \int \frac{P^{II} dz}{z} = -\int Q^{II} du, \quad Q^{III} = \int \frac{Q^{II} dz}{z} = \int P^{II} du.$$

$$P^{IIII} = \int \frac{P^{III} dz}{z} = -\int Q^{III} du, \quad Q^{IIII} = \int \frac{Q^{III} dz}{z} = \int P^{III} du.$$

etc.

etc.

etc.

etc.

etc.

etc.

vnde

vnde plures insignēs relationes deduci possunt.

§. 35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, vbi formulae integrales, in quibus sola z est variabilis, reducuntur ad alias formulas integrales, in quibus sola u est variabilis, cuiusmodi sunt, quae sequuntur

$$P^I = \int \frac{P dz}{z} = - \int Q du$$

$$P^{II} = \int \frac{dz}{z} \int \frac{P dz}{z} = - \int du \int P du$$

$$P^{III} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P dz}{z} = + \int du \int du \int Q du$$

$$P^{III} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P dz}{z} = + \int du \int du \int du \int P du$$

$$P^V = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P dz}{z} = - \int du \int du \int du \int du \int Q du$$

etc.

Similique modo pro altero genere

$$Q^I = \int \frac{Q dz}{z} = + \int P du$$

$$Q^{II} = \int \frac{dz}{z} \int \frac{Q dz}{z} = - \int du \int Q du$$

$$Q^{III} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q dz}{z} = - \int du \int du \int P du$$

$$Q^{III} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q dz}{z} = + \int du \int du \int du \int Q du$$

$$Q^V = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{Q dz}{z} = + \int du \int du \int du \int du \int P du$$

etc.

§. 36. Quod si iam nostrarum serierum, siue, quod eodem redit, quantitatum

$P, P^I, P^{II}, P^{III}, P^{IV}$ etc. et Q, Q^I, Q^{II}, Q^{III} etc.

cos tantum valores desideremus, quos adipiscuntur.

Tom. XIX. Nou. Comm.

M

posito

90 DE QUANTITATIBVS

posito $z = 1$, hoc commodi assequimur, vt in formulis integralibus vbi solus angulus u pro variabili habetur, statim ante integrationes ponere liceat $z = 1$, hoc autem factum erit

$$P = \frac{\cos u - 1}{z - 2 \cos u} = -\frac{1}{2} \quad \text{et} \quad Q = \frac{\sin u}{z - 2 \cos u} = \frac{1}{2} \cot \frac{1}{2} u$$

tum vero porro

$$\int P du = A - \frac{1}{2} u$$

$$\int du \int P du = B + A u - \frac{1}{4} u u$$

$$\int du \int du \int P du = C + B u + \frac{1}{2} A u u - \frac{1}{12} u^3$$

$$\int du \int du \int du \int P du = D + C u + \frac{1}{2} B u u + \frac{1}{6} A u^3 - \frac{1}{48} u^5$$

at pro formulis, vbi est q , calculus non tam concinne succedit; erit enim

$$Q = \frac{1}{2} \cot \frac{1}{2} u$$

$$\int Q du = l \sin \frac{1}{2} u$$

$$\int du \int Q du = \int du l \sin \frac{1}{2} u$$

quae formula cum omnem integrationem respuat, vix vltius progredi licet; interim tamen erit

$$\int du \int du \int Q du = \int du \int du l \sin \frac{1}{2} u$$

$$\int du \int du \int du \int Q du = \int du \int du \int du l \sin \frac{1}{2} u$$

§. 37. Quod ad priores formulas variabilem z inuoluentes attinet, per notas reductiones elicitur

$$\int \frac{P dz}{z} = \int \frac{dz}{z} \int \frac{P dz}{z} = l z \int \frac{P dz}{z} - \int \frac{P dz}{z} l z$$

vbi prius membrum $l z \int P dz$ euanescit posito $z = 1$, tum vero

$$\int \frac{dz}{z} \int \frac{P dz}{z} = \int \frac{dz}{z} \int \frac{dz}{z} \int \frac{P dz}{z} = + \int \frac{P dz}{z} \frac{(l z)^2}{2}$$

quibus expressionibus vltius exhibitis colligimus fore

$$P =$$

$$\begin{array}{l|l}
 P' = \int \frac{P dz}{z} & Q' = \int \frac{Q dz}{z} \\
 P'' = -\int \frac{P dz}{z} l z & Q'' = -\int \frac{Q dz}{z} l z \\
 P''' = +\int \frac{P dz}{z} \frac{(l z)^2}{1 \cdot 2} & Q''' = +\int \frac{Q dz}{z} \frac{(l z)^2}{1 \cdot 2} \\
 P^{IV} = -\int \frac{P dz}{z} \frac{(l z)^3}{1 \cdot 2 \cdot 3} & Q^{IV} = -\int \frac{Q dz}{z} \frac{(l z)^3}{1 \cdot 2 \cdot 3}
 \end{array}$$

§. 38. Ex his igitur fequentium formularum integralium valores assignare possumus casu quo $z=1$

$$\begin{aligned}
 P &= -\frac{1}{2} \\
 P' &= \int \frac{P dz}{z} = -l \sin. \frac{1}{2} u \\
 P'' &= -\int \frac{P dz}{z} l z = -B - A u + \frac{1}{4} u u \\
 P''' &= +\int \frac{P dz}{z} \frac{(l z)^2}{1 \cdot 2} = \int du \int du l \sin. \frac{1}{2} u \\
 P^{IV} &= -\int \frac{P dz}{z} \frac{(l z)^3}{1 \cdot 2 \cdot 3} = D + C u + \frac{1}{2} B u u + \frac{1}{4} A u^3 - \frac{1}{24} u^5 \\
 P^V &= +\int \frac{P dz}{z} \frac{(l z)^4}{1 \cdot 2 \cdot 3 \cdot 4} = \int du \int du \int du \int du l \sin. \frac{1}{2} u \\
 &\text{etc.}
 \end{aligned}$$

Eodemque modo

$$\begin{aligned}
 Q &= \frac{1}{2} \cot. \frac{1}{2} u \\
 Q' &= \int \frac{Q dz}{z} = A - \frac{1}{2} u \\
 Q'' &= -\int \frac{Q dz}{z} l z = -\int du l \sin. \frac{1}{2} u \\
 Q''' &= +\int \frac{Q dz}{z} \frac{(l z)^2}{1 \cdot 2} = -C - B u - \frac{1}{2} A u u + \frac{1}{24} u^3 \\
 Q^{IV} &= -\int \frac{Q dz}{z} \frac{(l z)^3}{1 \cdot 2 \cdot 3} = \int du \int du \int du l \sin. \frac{1}{2} u \\
 Q^V &= +\int \frac{Q dz}{z} \frac{(l z)^4}{1 \cdot 2 \cdot 3 \cdot 4} = E + D u + \frac{1}{2} C u u + \frac{1}{6} B u^3 + \frac{1}{24} A u^5 - \frac{1}{720} u^7 \\
 &\text{etc.}
 \end{aligned}$$

§. 39. Cum igitur fit

$$P = \frac{z \cos. u - z z}{1 - z z \cos. u + z z} \quad \text{et} \quad Q = \frac{z \sin. u}{1 - z z \cos. u + z z}$$

M 2

hacte-

haecenus id fumus affecuti, vt harum duarum formularum integralium

$$\int \frac{dz (\cos. u - z)}{1 - 2z \cos. u + z^2} (lz)^n \text{ et } \int \frac{dz \sin. u}{1 - 2z \cos. u + z^2} (lz)^n$$

valores casu $z = 1$ commodè per angulum u assignare valeamus, si modo constaret, quo facto quantitates A, B, C, D determinari oporteat, id quod vix alio modo nisi per ipsas series, vnde hae quantitates sunt natae, fieri posse videtur.

§. 40. Omissis igitur formulis integralibus, quae quantitatem Q inuoluunt, quippe quarum integratio minus succedit, alteras tantum consideremus, et posito statim $z = 1$ vbi fit $P = -\frac{1}{2}$ ita vt sit

$$\cos. u + \cos. 2u + \cos. 3u + \cos. 4u = -\frac{1}{2}$$

si per $d u$ multiplicemus et integremus, habebimus

$$Q = \frac{\sin. u}{1} + \frac{\sin. 2u}{2} + \frac{\sin. 3u}{3} + \frac{\sin. 4u}{4} + \frac{\sin. 5u}{5} \text{ etc.} = A - \frac{1}{2}u$$

quae constans nihilo aequalis videri potest, quia posito $u = 0$ summa seriei euanesce videtur; at sumto angulo u infinite paruo series praebebit

$$u + u + u + u + u + u + \text{etc. in infinitum}$$

notum autem est, talem seriem summam finitam habere posse, vnde hoc casu omissio statuamus $u = \pi$, seu potius $u = \pi + \omega$ prodibitque haec series existente ω angulo infinite paruo

$$-\omega + \omega - \omega + \omega - \omega + \omega - \omega + \text{etc.}$$

vbi quia signa alternantur, nullum est dubium, quin summa seriei euanescat, quae cum esse debeat $A - \frac{1}{2}u$ evidens

euidens est, fieri constantem $A = \frac{1}{2}\pi$, ita, vt iam habeamus

$$Q' = \frac{\sin. u}{1} + \frac{\sin. 2u}{2} + \frac{\sin. 3u}{3} + \frac{\sin. 4u}{4} + \frac{\sin. 5u}{5} \text{ etc.} = \frac{\pi - u}{2}.$$

Hoc modo constantem determinandi illustr. *Daniel Bernoulli* primus est vsus, qui praeterea multa praecleara circa indolem harum serierum annotauit.

§. 41. Multiplicemus porro hanc vltimam seriem per $-du$, et integratio dabit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{u^2}{4}$$

ad quam constantem inueniendam ponamus primo $u = 0$ fietque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \text{ etc.} = B$$

Cuius seriei summam iam pridem primus demonstrauit esse $= \frac{\pi^2}{6}$; verum si haec veritas nobis esset ignota, egregia illa methodo a magno *Bernoullio* adhibita vtamur ac ponamus $u = \pi$ eritque

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} \text{ etc.} = B - \frac{\pi\pi}{2} + \frac{\pi\pi}{4} = B - \frac{\pi\pi}{4}$$

ambae hae series additae dabunt

$$\frac{2}{1^2} - \frac{2}{4^2} + \frac{2}{9^2} - \frac{2}{16^2} \text{ etc.} = 2B - \frac{\pi\pi}{4}$$

cuius duplum praebet

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} = 4B - \frac{\pi\pi}{2} = B$$

vnde colligitur $B = \frac{\pi^2}{6}$ ita vt sit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} = \frac{\pi^2}{6} - \frac{\pi u}{2} + \frac{u^2}{4}.$$

§. 42. Eodem modo vltius progrediamur, et denuo per du multiplicando et integrando adipiscimur

M 3

$Q'' =$

$$Q^{III} = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} + \text{etc.} = C + \frac{\pi \pi u}{6} - \frac{\pi u u}{4} + \frac{\pi^3 u^3}{12}$$

vbi si statuatur $u = 0$, summa seriei manifesto eu-
nescit, prodiret enim posito $u = \omega$

$$\frac{\omega}{1^2} + \frac{\omega}{2^2} + \frac{\omega}{3^2} + \frac{\omega}{4^2} + \text{etc.} = \frac{\omega \pi \pi}{6}$$

quae ob $\omega = 0$ fit $= 0$ sicque erit $C = 0$ ideoque

$$Q^{III} = \frac{\sin. u}{1^3} + \frac{\sin. 2u}{2^3} + \frac{\sin. 3u}{3^3} + \frac{\sin. 4u}{4^3} + \text{etc.} = \frac{\pi \pi u}{6} - \frac{\pi u u}{4} + \frac{\pi^3 u^3}{12}$$

§. 43. Ducatur haec series in $-du$ et inte-
gratio praebit

$$p^{IV} = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} + \text{etc.} = D - \frac{\pi \pi u u}{12} + \frac{\pi u^3}{12} - \frac{\pi^4 u^4}{48}$$

hinc sumto $u = 0$ fiet

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = D$$

nunc vero fiat etiam $u = \pi$ fietque

$$-\frac{1}{1^4} + \frac{1}{2^4} - \frac{1}{3^4} + \frac{1}{4^4} - \frac{1}{5^4} + \text{etc.} = D - \frac{\pi^4}{48}$$

hae autem ambae series additae dant

$$\frac{2}{1^4} + \frac{2}{2^4} - \frac{2}{3^4} - \frac{2}{4^4} + \text{etc.} = 2D - \frac{\pi^4}{48}$$

quae octies sumta ut numeratores fiant $= 2^8$ prae-
bebit

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.} = 16D - \frac{\pi^4}{6}$$

unde oritur $D = \frac{\pi^4}{96}$ quae est eadem summa seriei

$$\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}$$

quam iam dudum inueneram, habebimus iam

$$P^{III} = \frac{\cos. u}{1^4} + \frac{\cos. 2u}{2^4} + \frac{\cos. 3u}{3^4} + \frac{\cos. 4u}{4^4} + \text{etc.} = \frac{\pi^4}{96} - \frac{\pi^2 u^2}{12} + \frac{\pi u^3}{12} - \frac{u^4}{48}$$

§. 44. Multiplicando iterum per du et inte-
grando consequimur

$$Q^V =$$

$$Q^V = \frac{\sin. u}{1^5} + \frac{\sin. 2 u}{2^5} + \frac{\sin. 3 u}{3^5} + \frac{\sin. 4 u}{4^5} + \text{etc.} = E + \frac{\pi^4 u}{90} - \frac{\pi^2 u^3}{36} + \frac{\pi u^5}{48} - \frac{u^7}{240}$$

vbi vti in casu penultimo constans E iterum fit = 0
ita vt habeamus

$$Q^V = \frac{\sin. u}{1^5} + \frac{\sin. 2 u}{2^5} + \frac{\sin. 3 u}{3^5} + \frac{\sin. 4 u}{4^5} \text{ etc.} = \frac{\pi u^4}{90} - \frac{\pi^2 u^2}{36} + \frac{\pi u^4}{48} - \frac{u^6}{240}$$

§. 45. Multiplicemus denuo per $-du$ proditque integrando

$$P^VI = \frac{\cos. u}{1^6} + \frac{\cos. 2 u}{2^6} + \frac{\cos. 3 u}{3^6} + \frac{\cos. 4 u}{4^6} \text{ etc.} = F - \frac{\pi^4}{90} \cdot \frac{u u'}{2} + \frac{\pi \pi^2}{6} \cdot \frac{u^3 u'}{24} - \frac{\pi^2}{5} \cdot \frac{u^5 u'}{120} + \frac{1}{2} \cdot \frac{u^6}{720}$$

vbi ad constantem determinandam ponatur $u=0$ eritque

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \text{etc.} = F$$

tum vero sumatur $u=\pi$ et fiet

$$-\frac{1}{1^6} + \frac{1}{2^6} - \frac{1}{3^6} + \frac{1}{4^6} - \text{etc.} = F - \frac{\pi^6}{480}$$

quae additae dant

$$\frac{2}{1^6} + \frac{2}{2^6} + \frac{2}{3^6} + \frac{2}{4^6} + \text{etc.} = 2F - \frac{\pi^6}{480}$$

quae multiplicetur per 32 vt omnes numeratores fiant 64 = 2⁶ et orietur

$$\frac{2}{1^6} + \frac{2}{2^6} + \frac{2}{3^6} + \frac{2}{4^6} + \text{etc.} = 64 F - \frac{\pi^6}{15} = F$$

vnde colligitur $F = \frac{\pi^6}{945}$ ita vt fit

$$P^VI = \frac{\cos. u}{1^6} + \frac{\cos. 2 u}{2^6} + \frac{\cos. 3 u}{3^6} + \frac{\cos. 4 u}{4^6} + \text{etc.} = \frac{\pi^6}{945} - \frac{\pi^4}{90} \cdot \frac{u^2}{2} + \frac{\pi^2}{6} \cdot \frac{u^4}{24} - \frac{\pi}{2} \cdot \frac{u^6}{720} + \frac{1}{2} \cdot \frac{u^8}{720}$$

§. 46. Has series vltius continuare superfluum foret, cum lex progressionis iam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim vsque ad potestatem trigesimam supputatas dedi. Quod, quo clarius perspiciatur istas summas sequenti modo repraesentemus

$$\frac{1}{1^2} +$$

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} \text{ etc.} &= \alpha \pi \pi \text{ vt fit } \alpha = \frac{\pi}{6} \\ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \text{etc.} &= \beta \pi^4 \text{ vt fit } \beta = \frac{\pi^4}{96} \\ \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} \text{ etc.} &= \gamma \pi^6 \text{ vt fit } \gamma = \frac{\pi^6}{945} \\ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \frac{1}{6^8} \text{ etc.} &= \delta \pi^8 \text{ vt fit } \delta = \frac{\pi^8}{9450} \\ \text{etc.} & \qquad \qquad \qquad \text{etc.} \qquad \qquad \qquad \text{etc.} \end{aligned}$$

atque his positis sequentes habebimus integrationes
pro casu scilicet $z = 1$

$$\begin{aligned} Q^I &= \int \frac{dz \sin u}{1 - 2z \cos u + z^2} = \frac{1}{2} \pi - \frac{1}{2} u = A \text{ tang. } \frac{\sin u}{1 - \cos u} \\ P^{II} &= - \int \frac{dz (\cos u - z)}{1 - 2z \cos u + z^2} \cdot \frac{1}{z} = \alpha \pi \pi - \frac{1}{2} \pi u + \frac{1}{2} \frac{u^2}{z} \\ Q^{III} &= + \int \frac{dz \sin u}{1 - 2z \cos u + z^2} \cdot \frac{(1z)^2}{2} = \alpha \pi \pi \frac{u}{2} - \frac{1}{2} \pi \cdot \frac{u^2}{2} + \frac{1}{2} \cdot \frac{u^3}{6} \\ P^{IV} &= - \int \frac{dz (\cos u - z)}{1 - 2z \cos u + z^2} \cdot \frac{(1z)^3}{6} = \beta \pi^4 - \alpha \pi \pi \cdot \frac{u^2}{2} + \frac{1}{2} \pi \cdot \frac{u^3}{6} - \frac{1}{2} \cdot \frac{u^4}{24} \\ Q^V &= + \int \frac{dz \sin u}{1 - 2z \cos u + z^2} \cdot \frac{(1z)^4}{24} = \beta \pi^4 \cdot \frac{u}{2} - \alpha \pi \pi \cdot \frac{u^3}{6} + \frac{1}{2} \pi \cdot \frac{u^4}{24} - \frac{1}{2} \cdot \frac{u^5}{120} \\ P^{VI} &= - \int \frac{dz (\cos u - z)}{1 - 2z \cos u + z^2} \cdot \frac{(1z)^5}{120} = \gamma \pi^6 - \beta \pi^4 \cdot \frac{u^2}{2} + \alpha \pi \pi \cdot \frac{u^4}{24} - \frac{1}{2} \pi \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720} \\ Q^{VII} &= + \int \frac{dz \sin u}{1 - 2z \cos u + z^2} \cdot \frac{(1z)^6}{720} = \gamma \pi^6 \cdot \frac{u}{2} - \beta \pi^4 \cdot \frac{u^3}{6} + \alpha \pi \pi \cdot \frac{u^5}{120} - \frac{1}{2} \pi \cdot \frac{u^6}{720} + \frac{1}{2} \cdot \frac{u^7}{5040} \\ \text{etc.} & \qquad \qquad \qquad \text{etc.} \qquad \qquad \qquad \text{etc.} \end{aligned}$$

§. 47. Operae pretium erit, aliquos casus, quibus angulo u datus valor tribuitur, ob oculos exponere. Ponamus igitur $u = 0$ quo casu formulae nostrae alternatim euanescent, reliquae vero praebunt

$$\begin{aligned} - \int \frac{dz}{1-z} l z &= \alpha \pi \pi = \frac{\pi \pi}{6} \\ - \int \frac{dz}{1-z} \frac{(1z)^3}{6} &= \beta \pi^4 = \frac{\pi^4}{96} \\ - \int \frac{dz}{1-z} \frac{(1z)^5}{120} &= \gamma \pi^6 = \frac{\pi^6}{945} \end{aligned}$$

his affines sunt formulae, quae oriuntur ex positione

$$u = \pi$$

$u = \pi$ vbi iterum abeunt alternae sinum u inuolventes et remanebunt sequentes

$$\int \frac{dz}{1+z} l z = -\frac{\pi \pi}{12} = -\frac{1}{2} \alpha \pi \pi$$

$$\int \frac{dz}{1+z} \frac{(l z)^3}{6} = -\frac{7 \pi^4}{720} = -\frac{7}{8} \beta \pi^4$$

$$\int \frac{dz}{1+z} \frac{(l z)^5}{320} = -\frac{31}{82} \gamma \pi^6$$

$$\int \frac{dz}{1+z} \frac{(l z)^7}{720} = -\frac{127}{128} \delta \pi^8$$

§. 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam euanescent posito $u = \pi$; deinde non minus notatu dignum est, easdem formulas quoque euanescere posito $u = 2\pi$, sola prima excepta, quippe quae etiam non euanescit posito $u = 0$; reliquae vero, scilicet tertia, quinta, septima etc. certe euanescent casibus $u = 0$ et $u = \pi$, quin etiam $u = 2\pi$. Quod quo clarius appareat, has formulas per factores repraesentemus eritque tertiae valor

$$= \frac{1}{22} u (\pi - u) (2\pi - u),$$

quintae vero valor reperitur

$$\frac{2}{125} (\pi - u) (2\pi - u) (4\pi \pi + 6\pi u - 3u u),$$

quod etiam in sequentibus vsu venit. In genere autem obseruari meretur, omnes nostras formulas sola prima excerpta eosdem sortiri valores, siue ponatur $u = 0$ siue $u = 2\pi$, quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensum locum habere debere si ponatur $u = 4\pi$ et $u = 6\pi$, verum Illustr. Bernoullius iam luculenter ostendit, angulum u in his valoribus non ultra

quatuor rectos augeri posse. Huiusmodi autem anomaliam etiam in omnibus vulgaribus seriebus quibus arcus exprimuntur occurrit, atque adeo in *Leibniziana*, in qua est

$$u = \frac{\text{tang. } u}{1} - \frac{(\text{tang. } u)^3}{3} + \frac{(\text{tang. } u)^5}{5} - \frac{(\text{tang. } u)^7}{7} + \frac{(\text{tang. } u)^9}{9} - \text{etc.}$$

angulum u non ultra 180 gr. augere licet. Si enim poneremus $u = 180^\circ + u$ foret utique $\text{tang. } u = \text{tang. } u$ neque tamen series illa exprimeret arcum $\pi + u$ sed tantum arcum u , cuiusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u = 0$ denominator fiat $(1 - z)$ qui casu $z = 1$ evanescit ideoque formula in infinitum excrecit, id quod in sequentibus, quae per $1z$ sunt multiplicatae, non amplius evenit, quia $\frac{1z}{1-z}$ casu $z = 1$ non amplius fit infinitus sed tantum $= -1$ et si maior potestas logarithmi adsit fit adeo $= 0$.

§. 49. Ponamus nunc etiam $u = 90^\circ$, seu $u = \frac{\pi}{2}$, ut sit $\cos u = 0$ et $\sin u = 1$ hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int \frac{dz}{1+z^2} = \frac{\pi}{4}$$

$$\int \frac{z dz}{1+z^2} \log z = -\frac{\pi \pi}{32}$$

$$\int \frac{dz}{1+z^2} \frac{(1z)^2}{2} = \frac{\pi^3}{32}$$

$$\int \frac{z dz}{1+z^2} \frac{(1z)^3}{6} = -\frac{\pi^4}{960}$$

§. 50. Consideremus etiam casum $u = 60^\circ$ gr. huc $u = \frac{\pi}{3}$ vt fit $\cos. u = \frac{1}{2}$ et $\sin. u = \frac{\sqrt{3}}{2}$ et formulae generales perducent ad sequentia integralia

$$\frac{\sqrt{3}}{2} \int \frac{dz}{1-z+zz} = \frac{\pi}{3}$$

$$-\frac{1}{2} \int \frac{dz(1-z)}{1-z+zz} / z = -\frac{\pi \pi}{36}$$

$$\frac{\sqrt{3}}{2} \int \frac{dz}{1-z+zz} \frac{(1-z)^2}{z} = \frac{5\pi^2}{162}$$

Simili modo si ponamus $u = 120^\circ = \frac{2\pi}{3}$ vt fit $\cos. u = -\frac{1}{2}$ et $\sin. u = \frac{\sqrt{3}}{2}$ sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2} \int \frac{dz}{1+z+zz} = \frac{\pi}{6}$$

$$\frac{1}{2} \int \frac{dz(1+z)}{1+z+zz} / z = -\frac{\pi \pi}{144}$$

$$\frac{\sqrt{3}}{2} \int \frac{dz}{1+z+zz} \frac{(1+z)^2}{z} = \frac{2\pi^2}{81}$$

sicque pro lubitu numerus huiusmodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z=1$ sunt deductae, ita eodem modo alteram seriem Q pertrahemus. Cum igitur sit

$$Q = \sin. u + \sin. 2u + \sin. 3u + \sin. 4u \text{ etc.} = \frac{1}{2} \cot. \frac{1}{2} u$$

si per $-du$ multiplicemus et integremus, reperitur series

$$P = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = \frac{1}{2} - \int \sin. \frac{1}{2} u + A$$

pro qua constante determinanda ponatur $u = \pi$ vt fit

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \text{etc.} = A$$

quocirca fit $A = -1/2$ ita, vt habeamus

$$P = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = -1/2 \sin. \frac{1}{2} u$$

N 2

pro

pro quo valore scribamus breuitatis gratia $u \Delta : u$ si quidem eum spectamus tanquam certam ipsius u functionem ita vt sit $P^I = \Delta : u$.

§. 52. Multiplicando porro per du et integrando nanciscimur hanc seriem

$$Q^{II} = \frac{\sin. u}{1^2} + \frac{\sin. 2u}{2^2} + \frac{\sin. 3u}{3^2} + \frac{\sin. 4u}{4^2} \text{ etc. } = \int du \Delta : u = \Delta^I : u$$

vbi haec formula integralis inuoluet certam constantem quam facile definire licet ex casu $u = 0$ quia enim series euanescit, fieri debet $\Delta^I : 0 = 0$ sicque integratio plene determinatur.

§. 53. Si eodem modo vltcrius progrediamur multiplicando per $-du$, prodibit haec series

$$P^{III} = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} = -\int du \Delta'' : u = \Delta'' : u$$

Iam ad constantem, quae in hac expressione continetur, definiendam sit $1^\circ u = 0$ eritque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \text{ etc. } = \Delta'' : 0 \quad 1^\circ. \text{ sit } u = \pi \text{ et fiet}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} \text{ etc. } = \Delta'' : \pi \text{ quibus additis prodit}$$

$$\frac{2}{3^2} + \frac{2}{4^2} + \frac{2}{5^2} + \frac{2}{6^2} + \text{ etc. } = \Delta'' : 0 + \Delta'' : \pi \text{ hacque quatuor sumta erit.}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{ etc. } = 4 \Delta'' : 0 + 4 \Delta'' : \pi = \Delta'' : 0 \text{ vnde oritur}$$

$$3 \Delta'' : 0 + 4 \Delta'' : \pi = 0$$

ex qua constans in formulam nostram integram

$$\Delta'' : u = -\int du \Delta^I u$$

ingressa determinari debet.

§. 54. Multiplicemus denuo per du et integremus prodibitque

$$Q^{IV} = \frac{\sin. u}{1^4} + \frac{\sin. 2u}{2^4} + \frac{\sin. 3u}{3^4} + \frac{\sin. 4u}{4^4} + \text{ etc. } = \int du \Delta'' : u = \Delta''' : u$$

atque

atque haec functio $\Delta''' : u$ ita debet determinari, vt
euanescat sumto $u = 0$ siue vt fiat $\Delta''' : 0 = 0$.
Eodem modo vltcrius progrediendo fiet

$$p^v = \frac{\cos. u}{1^5} + \frac{\cos. 2u}{2^5} + \frac{\cos. 3u}{3^5} + \frac{\cos. 4u}{4^5} = -\int du \Delta''' : u = \Delta''' : u$$

huiusque functionis indoles sequenti modo determi-
nabitur: ponatur scilicet vt haftenus $u = 0$ et $u = \pi$
eritque

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \Delta^{IV} : 0 \text{ et}$$

$$-\frac{1}{1^5} + \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} + \text{etc.} = \Delta^{IV} : \pi \text{ hinc addendo}$$

$$\frac{2}{2^5} + \frac{2}{4^5} + \frac{2}{6^5} + \frac{2}{8^5} + \text{etc.} = \Delta^{IV} : 0 + \Delta^{IV} : \pi \text{ et multiplicando per } 16$$

$$\frac{2}{1^5} + \frac{2}{2^5} + \frac{2}{3^5} + \frac{2}{4^5} + \text{etc.} = 16 \Delta^{IV} : 0 + 16 \Delta^{IV} : \pi = \Delta^{IV} : 0$$

$$\text{sicque fieri debebit } 15 \Delta^{IV} : 0 + 16 \Delta^{IV} : \pi = 0 \text{ etc.}$$

§. 55. Hinc igitur sequentes adipiscemur in-
tegrationes pro casu $z = 1$

$$\text{I. } -\int \frac{dz (\cos. u - z)}{1 - 2z \cos. u + z^2} = -l z \sin. \frac{1}{2} u = \Delta : u$$

$$\text{II. } \int \frac{dz \sin. u}{1 - 2z \cos. u + z^2} l z = \int du \Delta u = \Delta' : u$$

$$\text{III. } -\int \frac{dz (\cos. u - z)}{1 - 2z \cos. u + z^2} \frac{(l z)^2}{2} = -\int du \Delta' u = \Delta'' : u$$

$$\text{IV. } \int \frac{dz \sin. u}{1 - 2z \cos. u + z^2} \frac{(l z)^3}{6} = \int du \Delta'' u = \Delta''' u$$

$$\text{V. } -\int \frac{dz (\cos. u - z)}{1 - 2z \cos. u + z^2} \frac{(l z)^4}{24} = -\int du \Delta''' u = \Delta^{IV} u$$

$$\text{VI. } \int \frac{dz \sin. u}{1 - 2z \cos. u + z^2} \frac{(l z)^5}{120} = \int du \Delta^{IV} u = \Delta^V u$$

etc.

etc.

etc.

etc.

Has autem expressiones facile quousque libuerit con-
tinuare licet, si modo integratio cuiusque integralis

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rite instituat^r; conditiones autem, quas impleri oportet, sequenti modo referri possunt

$\Delta^I : 0 = 0$	$3 \Delta^{II} : 0 + 4 \Delta^{II} : \pi = 0$
$\Delta^{III} : 0 = 0$	$15 \Delta^{IV} : 0 + 16 \Delta^{IV} : \pi = 0$
$\Delta^V : 0 = 0$	$63 \Delta^{VI} : 0 + 64 \Delta^{VI} : \pi = 0$
$\Delta^{VII} : 0 = 0$	$255 \Delta^{VIII} : 0 + 255 \Delta^{VIII} : \pi = 0$
etc. etc.	etc. etc.

caeterum quia posteriores integrationes absolvere non licet, hinc parum vtilitatis exspectare possumus.

§. 56. Caeterum methodus, qua hic sumus vsi, ad constantes per quamque integrationem ingressas determinandas, a celeberrimo *Bernoullio* primum est adhibita atque eo maiori attentione digna est aestimanda, quod eius ope summationes meae serierum reciprocarum potestatum obtineri possunt, quandoquidem credideram, eas non aliter nisi ex consideratione infinitorum arcuum qui vel eodem sinu vel cosinu gaudent, demonstrari posse.